Math 245C Lecture 26 Notes

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1 Introduction to Sobolev Spaces

1.1 Sobolev spaces and uniqueness of distributional derivatives

Throughout this section, $\Omega \subseteq \mathbb{R}^d$ is a nonempty, open set.

Proposition 1.1. *Let $f \in L^1_{\mathrm{loc}}(\Omega)$ be such that $\int_\Omega f\phi\, dx = 0$ for all $\phi \in C_c^\infty(\Omega)$. Then $f \equiv 0$ a.e.*

Proof. Let $\rho \in C_c^\infty(\mathbb{R}^d)$ be such that $\rho \geq 0$, $\int_{\mathbb{R}^d} \rho\, dx = 1$, and $\operatorname{supp}(\rho) = B_1(0)$. Set $\rho_\varepsilon(x) = \varepsilon^{-d}\rho(x/\varepsilon)$. Let $x \in U$, and let $0\varepsilon_0 < \operatorname{dist}(x, \partial\Omega)$. Then

$$\rho_\varepsilon * f(x) = \int_{B_\varepsilon(x)} \rho_\varepsilon(x-y) f(y)\, dy = 0, \qquad 0 < \varepsilon < \varepsilon_0.$$

Thus, for almost every x,

$$0 = f(x) = \lim_{\varepsilon \to 0} \rho_\varepsilon * f(x). \qquad \square$$

Definition 1.1. Let $1 \leq p \leq \infty$, and let $m \in \mathbb{N}$. We say that $f \in W^{p,m}_{\mathrm{loc}}(\Omega)$ if $f \in L^p_{\mathrm{loc}}(\Omega)$ and if for every multi-index $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq m$, there exists $g_\alpha \in L^p_{\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega f \partial^\alpha \phi\, dx = (-1)^{|\alpha|} \int_\Omega g_\alpha \phi\, dx \qquad \forall \phi \in C_c^\infty(\Omega).$$

In other words, the distributional derivative $\partial^\alpha f \in L^p_{\mathrm{loc}}$. When $f \in L^p(\Omega)$ and $g_\alpha \in L^p(\Omega)$ for $|\alpha| \leq m$, we write $f \in W^{m,p}(\Omega)$.

Remark 1.1. Thanks to the previous proposition, when g_α exists, it is uniquely determined a.e.

1.2 Translation of distributions

Notation: Let $\phi \in C_c^\infty(\Omega)$, and let $y \in \mathbb{R}^d$. We set $\phi_y(x) = \phi(x-y) = (\tau_y \phi)(x)$. Note that $\operatorname{supp}(\varphi_y) = \operatorname{supp}(\phi) + y$. Set

$$O_\phi = \{y \in \mathbb{R}^d : y + \operatorname{supp}(\phi) \subseteq \Omega\} = \{y \in \mathbb{R}^d : \operatorname{supp}(\phi_y) \subseteq \Omega\}.$$

Proposition 1.2. *O_ϕ is open and nonempty.*

Proof. Let $y \in O_\phi$, and set $\delta = \operatorname{dist}(y + \operatorname{supp}(\phi), \Omega^c) > 0$. If $y \in O_\phi$, then $B_{\delta/2}(y) \subseteq O_\phi$. Hence, O_ϕ is open. $O_\phi \neq \varnothing$ because $0 \in O_\phi$. $\square$

Proposition 1.3. *If $T \in \mathcal{D}'(\Omega)$, $y \mapsto T(\phi_y)$ is continuous.*

Proof. Let $(y_n)_n \subseteq O_\phi$ be a sequence converging to y. We are to show that $\lim_n T(\phi_{y_n}) = T(\phi_y)$. Note that

$$\phi_{y_n}(x) = \phi(x - y_n) = \phi(x-y) - \int_0^1 \nabla\phi(x - y + t(y_n - y)) \cdot (y_n - y)\, dt.$$

This gives us that $(\phi_{y_n})_n$ converges to ϕ_y in C_c^∞. Indeed,

$$|\partial^\alpha \phi_{y-N} - \partial^\alpha \phi_y| \leq \|\nabla \partial^\alpha \phi\|_\infty \|y_n - y\|.$$

Since T is continuous, we conclude that

$$\lim_n T(\phi_{y_n}) = T(\phi_y).$$

$\square$

Theorem 1.1. *Let $\phi \in C_c^\infty(\Omega)$, and let $T \in \mathcal{D}'(\Omega)$. Set $f(y) = T(\phi_y)$ for $y \in O_\phi$.*

1. *$f \in C^\infty(O_\phi)$, and*
$$D^\alpha f(y) = (-1)^{|\alpha|} T((D^\alpha \phi)_y).$$

2. *If $\psi \in L^1(O_\phi)$ has compact support, then*
$$T(\psi * \phi) = \int_{O_\phi} \psi(y) f(y)\, dy.$$

Proof. One proves by induction on α that $\partial^\alpha f$ exists, is continuous, and satisfies the equation. Assume $|\alpha| = 1$. Let $e_1, \dots, e_d$ be the standard basis of $\mathbb{R}^n$. We have for $t \in \mathbb{R}$

$$\phi_{y+te_i}(x) = \phi(x - y - te_i) = \phi(x-y) - \int_0^1 \partial_i \phi(x - y - t\tau e_i)\, d\tau.$$

Hence,

$$\frac{\phi_{y+te_i}(x) - \phi_y(x)}{t} = -\int_0^1 \partial_i \phi(x - y - t\tau e_i)\, d\tau.$$

In fact, we have

$$\frac{\partial^\alpha \phi_{y+te_i}(x) - \partial^\alpha \phi_y(x)}{t} = -\int_0^1 [\partial^\alpha \partial_i \phi(x - y - t\tau e_i) - \partial^\alpha \partial_i \phi(x-y)]\, d\tau - \partial^\alpha \partial_i \phi(x-y).$$

This shows that

$$\frac{\phi_y + te_u - \phi_y}{t}(x) \to -\partial_i \phi(x-y)$$

pointwise and in $C_c^\infty(\Omega)$. Hence,

$$\lim_{t\to 0} \frac{f(y+te_i) - f(y)}{t} = \lim_{t\to 0} \frac{T(\phi_y + te_i) - T(\phi_y)}{t} = \lim_{t\to\infty} T\left(\frac{\phi_{y+te_i} - \phi_y}{t}\right) = T(-(\partial\phi(x))_y).$$

Since $\partial_i \phi \in C_c^\infty(\Omega)$, by the previous proposition, $y \to T((\partial_i\phi)_y)$ is continuous. In conlcusion, f is continuously differentiable, and $\nabla d(y) = -T((\nabla\phi)_y)$. This concludes the proof of the first statement when $|\alpha| = 1$. By induction, we obtain the result for all α. $\square$

We will prove the second statement next time.